Lévy random walks with fluctuating step number and multiscale behavior

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Random walks with step number fluctuations are examined in *n* dimensions for when step lengths comprising the walk are governed by stable distributions, or by random variables having power-law tails. When the number of steps taken in the walk is large and uncorrelated, the conditions of the Lévy-Gnedenko generalization of the central limit theorem obtain. When the number of steps is correlated, infinitely divisible limiting distributions result that can have Lévy-like behavior in their tails but can exhibit a different power law at small scales. For the special case of individual steps in the walk being Gaussian distributed, the infinitely divisible class of *K* distributions result. The convergence to limiting distributions is investigated and shown to be ultraslow. Random walks formed from a finite number of steps modify the behavior and naturally produce an inner scale. The single class of distributions derived have as special cases, *K* distributions, stable distributions, distributions with power-law tails, and those characteristic of high and low frequency cascades. The results are compared with cellular automata simulations that are claimed to be paradigmatic of self-organized critical systems. [S1063-651X(99)11511-3]

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I. INTRODUCTION

Multidisciplinary interest in self-organized critical (SOC) states [1] continues to burgeon (see, e.g., Ref. [2]). The concept of SOC refers to the spontaneous emergence of temporal and spatial scale invariance in nonequilibrium systems. Consideration of the non-Gaussian stochastic processes that are capable of describing SOC states is an important allied topic. A random process that has become a topical candidate is that described by the stable Lévy class [3], whose distributions possess power-law tails and whose integral moments typically diverge. Examples of behavior that are claimed to be relevant to the physical sciences include transport of matter and the concomitant emission of radiation in astrophysical accretion disc systems [4], the Earth's magnetosphere during geomagnetic substorms [5], transport regimes near the edge regions of magnetically confined plasmas [6], Taylor-Couette flow [7], the scattering of radiation by random phase screens and rough surfaces [8], and particle [9,10] and energy [11] transport within piles of granular material. The "sandpile" paradigm has since become a canonical exemplar of SOC. Biological behaviors include those cited in Ref. [12], together with the errant feeding patterns adopted by albatrosses [13]. Evidence also exists to suggest that the price of equities can be subject to Lévy-like fluctuations on an intraday basis [14], and that the economic growth of many industrialized nations is similarly non-Gaussian over considerably longer epochs [15]. This list is by no means exhaustive or entirely representative, but does serve to illustrate the diversity of systems that can be described by scale-invariant random processes. The manifold occurrences of Lévy-like processes in nature has prompted workers to seek an underlying cause for their prevalence and an information theory based approach has predicted distributions whose asymptotic behavior matches with a Lévy distribution [16].

The class of Lévy distributions is defined through their characteristic function $C(u)$, which in its most general form may be written as:

$$\ln C(u) = -|u|^{\gamma}(1 + i\beta\omega(\gamma,u)),$$

where the characteristic index $0<\gamma\leqslant 2$ governs the form of the tail of the distribution, the index $-1\leqslant\beta\leqslant 1$ determines the symmetry properties of the distribution, and $\omega(\gamma,u) = \mathrm{sgn}(u)\tan(\gamma\pi/2)$ if $\gamma\neq 1$ and $(2/\pi)\ln|u|$ if $\gamma=1$. The probability density function (PDF) is obtained through Fourier inversion of $C(u)$:

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(u)\exp(iux)\,du.$$

If $\beta>0$, the distribution is defined for positive values of x, the converse is true for $\beta<0$, and $\beta=0$ describes symmetric distributions. Of these latter types, well-known closed form expressions for $p(x)$ exist for the cases $\gamma=2$ (Gaussian distribution) and $\gamma=1$ (Cauchy distribution). However, other less well documented analytical forms exist for special cases [17] and are worthy of reintroducing to the literature; e.g., for $\gamma=\frac{2}{3}$, the symmetric distribution is:

$$p(x) = \left(\frac{3}{\pi x^2}\right)^{1/2}\exp\left(-\frac{2}{27x^2}\right)W_{1/2,1/6}\left(\frac{4}{27x^2}\right),$$

where $W_{a,b}(x)$ is the Whittaker function [18], and for $\gamma=\frac{1}{2}$ the density is expressible in terms of the Fresnel integrals $C(x)$, and $S(x)$ [18]:

$$p(x) = \frac{1}{2x\xi}[\cos(1/4x)(1-2C(\xi)) + \sin(1/4x)(1-2S(\xi))],$$

with $\xi = 1/(2\pi x)^{1/2}$. Closed form expressions can also be obtained when $\beta=1$, in which case $\gamma=1/2$ yields

$$p(x)=\frac{1}{\sqrt{2\pi x^3}}\exp\left(-\frac{1}{2x}\right),\qquad x>0,$$

and $\gamma=1/3$ gives

$$p(x)=\frac{1}{\pi}\left(\frac{2}{3^{7/6}x}\right)^{3/2}K_{1/3}\left(\frac{4}{3^{7/4}}\left(\frac{2}{3x}\right)^{1/2}\right),\qquad x>0,$$

with $K_\nu(x)$ a modified Bessel function [18]. All these distributions have an asymptotic form for large x that is a power law with $p(x)\sim x^{-1-\gamma}$ to leading order. The epithet ''stable'' refers to the property that the distributions are invariant under convolution with themselves, revealing their importance to random walks, for if N Lévy distributed random variables are added the distribution of the resultant is also a Lévy distribution. This illustrates that if the distribution of a physical property is stable, the statistics of that property will be persistent.

The Tsallis entropy [19] is a generalization of the familiar Maxwell-Boltzmann entropy, or Shannon information [20], and is related to the information dimension that is used extensively to describe multifractal structures such as those resulting from diffusion limited aggregation, for example [21]. It is parametrized through a real number q, whose value places weight upon the probability of occurrence of small- or large-scale events depending upon whether q is greater or less than unity. The Tsallis entropy provides a formalism that allows generalization of most of the principal results of thermodynamics and statistical physics, viz. the Boltzmann H theorem [22], the Onsager reciprocity theorem [23], the fluctuation-dissipation theorem [24] and the concavity of the entropy for $q>0$ [25]. However, there is, apparently, no basic equation akin to Liouville's theorem from which it can be obtained. Varying the Tsallis entropy subject to imposing two constraints, that of unit normalization and the (suitably defined) variance [16], obtains distributions that can possess power-law tails depending upon the value of q. The calculation described in Ref. [16] determines the canonical distribution function.

The principal purpose of this paper is to consider the grand canonical ensemble through determining the distribution for the resultant of an n-dimensional random walk for when the steps comprising the walk fluctuate in number and when the length of each step is governed by a stable distribution, or by distributions that are asymptotic to stable distributions. The motivation for such a development is prompted by past analyses of random walks with fluctuating step number which have proved to be an effective tool for studying the clutter in coherent optical and microwave remote sensing systems [26], for example. These provide a graphic illustration of the failure of the central limit theorem of classical statistics [27], and a tangible method for producing non-Gaussian statistics through a physical model. An example that can be derived in this fashion is the K distribution [28], and it will be shown how this results as a limiting special case of a Lévy random walk with a fluctuating number of steps. Step number fluctuations introduce correlations into the random walk, with fluctuations on short scales modulated by larger scale structures. These correlations lead to a class of infinitely divisible limiting distributions involving two parameters. These distributions are shown to have special cases embracing K distributions, Lévy distributions, distributions possessing power-law tails, and those characteristic of high- and low-frequency cascades. Each of these regimes will be explored.

The paper is organized as follows. Section II considers the statistics of random walks having step lengths governed by stable distributions, or by distributions possessing power-law tails for when the number of steps in the walk is itself a random variable. Particular attention is paid to negative-binomial number fluctuations that can account for correlations or clustering in the number of steps. Limiting forms of the distributions are obtained for when the *average* number of steps in the walk $\overline{N}\to\infty$, and solutions in terms of tabulated functions are obtained for special cases. The general form that these distributions adopt at small- and large-scale sizes is studied. Section III considers how these distributions are modified for when the random walks comprise a *finite* average number of steps and the convergence of these distributions to their limiting forms. Section IV discusses ways of characterizing differences between these distributions and Lévy distributions using fractional moments. Section V highlights the significance of the various parameter regimes that describe these distributions, and discusses their relationship with multicascade behavior and self-organized critical systems. Section VI summarizes results and draws conclusions. Many of the technical details are assigned to appendices. Appendix A generalizes the results in Ref. [16] by obtaining the distribution of an n-dimensional random variable x through variation of the Tsallis entropy subject to constraining the renormalized rth moment, where r may be fractional [29]. This generalization extends the parameter space in which random processes that are asymptotic to Lévy processes can occur. Other details pertaining to the analysis of the grand canonical ensemble are assigned to other appendices.

II. RANDOM WALKS WITH FLUCTUATING STEP NUMBER

This section derives the probability density function (PDF) for the resultant of a random variable $\mathbf{x}$ in n dimensions. Here $\mathbf{x}$ is an n vector of distance $x=|\mathbf{x}|$ from the origin having the remaining $n-1$ polar components distributed uniformly over the surface of a unit hypersphere. The PDF for the resultant of a coherent addition of N such random variables,

$$\mathbf{x}=\sum_{j=1}^{N}\mathbf{x}_j,$$

is facilitated through working with the characteristic function of the random variables $\mathbf{x}$. When each of the $\mathbf{x}_j$ is statistically similar and independent, the characteristic function of $\mathbf{x}$ is the N-fold product of the characteristic functions for the $\mathbf{x}_j$, so that

$$C_N(\mathbf{u})=(C(\mathbf{u}))^N,$$

where

$$C(\mathbf{u})=\langle\exp(-i\mathbf{u}\cdot\mathbf{x})\rangle=\frac{(2\pi)^{n/2}}{u^{n/2-1}}\int_0^\infty x^{n/2}p(x)J_{n/2-1}(xu)dx, \tag{1}$$

with $J_n(x)$ a Bessel function of the first kind [18], $u=|\mathbf{u}|$. The PDF of $\mathbf{x}$ then follows by Fourier inversion of Eq. (1),

$$p_N(\mathbf{x})=\frac{1}{(2\pi)^n}\int \exp(i\mathbf{x}\cdot\mathbf{u})C_N(\mathbf{u})d\mathbf{u}, \tag{2}$$

where the subscript N has been introduced to denote a random walk comprising N steps. The length of each step in the random walk is governed by a Lévy distribution, or one whose properties are asymptotic to a Lévy distribution, as discussed in Appendix A. In these circumstances the characteristic function may be taken to have the form

$$C_N(\mathbf{u})=\exp(-\Lambda|u|^\gamma),$$

where $0<\gamma\leqslant 2$.

To treat the grand canonical ensemble requires consideration of a random walk whose individual steps are described by stable distributions but where the number of steps in the walk fluctuate. Suppose that the discrete random variable N has probability distribution function $P(N)$. The characteristic function that results from averaging over all realizations of N is

$$C_{\bar{N}}(u)=\sum_N P(N)(C(u))^N.$$

To enable further progress requires a model for the number fluctuations. Taking $P(N)$ to be the negative binomial distribution accounts for clustering in the number of steps and is the steady state process that describes a birth-death-immigration process [28,30,31]. In this case,

$$P(N)=\binom{N+\alpha-1}{N}\frac{(\bar{N}/\alpha)^N}{(1+\bar{N}/\alpha)^{N+\alpha}},$$

where $\bar{N}$ is the mean of the distribution and $\alpha>0$ is the cluster parameter. The cluster parameter is related to the variance of the negative binomial distribution through $\mathrm{Var}(N)/\bar{N}^2=1/\alpha$, which is the ratio of immigration to birth rate in the birth-death-immigration process [28]. When $\alpha=1$ the negative binomial distribution is the Bose-Einstein or geometric distribution, and accounts for thermal number fluctuations. When $\alpha\to\infty$ it is the Poisson distribution, describing uncorrelated steps. Using this model gives the averaged characteristic function to be

$$C_{\bar{N}}(u)=\left(1+\frac{\bar{N}}{\alpha}(1-\exp(-\Lambda u^\gamma))\right)^{-\alpha}, \tag{3}$$

whereupon the PDF is

$$p_{\bar{N}}(x)=\frac{2^{1-n/2}}{\Gamma(n/2)}x^{n/2}\int_0^\infty u^{n/2}J_{n/2-1}(xu)\times\left(1+\frac{\bar{N}}{\alpha}(1-\exp(-\Lambda u^\gamma))\right)^{-\alpha}du. \tag{4}$$

Rescaling the characteristic function through the transformation $u\to u/\bar{N}^{1/\gamma}$ and then letting $\bar{N}\to\infty$, one obtains the limit distribution for $p_{\bar{N}}(x)$:

$$p_\infty(x)=\frac{2^{1-n/2}}{\Gamma(n/2)}x^{n/2}\int_0^\infty u^{n/2}J_{n/2-1}(xu)(1+\Lambda u^\gamma/\alpha)^{-\alpha}du. \tag{5}$$

It should be stressed that this result is valid if the individual steps in the walk are drawn from the class of stable distributions or indeed from those described by equations appearing in Appendix A which are asymptotic to stable distributions. Note that the further transformations $u\to(1/\Lambda)^{1/\gamma}u$, followed by $x\to(\Lambda)^{1/\gamma}x$, yield the compact result for the scaled PDF:

$$p_\infty(x)=\frac{2^{1-n/2}}{\Gamma(n/2)}x^{n/2}\int_0^\infty u^{n/2}J_{n/2-1}(xu)(1+u^\gamma/\alpha)^{-\alpha}du. \tag{6}$$

While it is not possible to evaluate Eqs. (4)–(6) for arbitrary values of γ and α, some special cases that can be determined exactly are worthy of mention.

When $\alpha\to\infty$ in Eq. (6), the characteristic function $\to\exp(-u^\gamma)$, which retrieves the Lévy distribution. Hence for Poisson number fluctuations, the Lévy-Gnedenko generalization [32] of the central limit theorem applies. If $\gamma=2$ and α is arbitrary, then Eq. (6) can be evaluated exactly to obtain

$$p(x)=\frac{2\alpha^{1/2}}{\Gamma(n/2)\Gamma(\alpha)}\left(\frac{\alpha^{1/2}x}{2}\right)^{n/2+\alpha-1}K_{\alpha-n/2}(\alpha^{1/2}x), \tag{7}$$

where $K_\nu(x)$ is a modified Bessel function [18] from which the eponymous distribution derives its name. The K distribution does not possess a power-law tail, rather, for $x\gg1$, it is asymptotic to the Γ distribution with $p(x)\sim x^{(n-3)/2+\alpha}\times\exp(-x)$. All its moments exist and are in excess of the equivalent order moments of a Gaussian distribution. Expressions for Eqs. (5) and (6) in closed form or involving tabulated functions cannot be found for arbitrary values of α and γ, nevertheless, the behavior for both large and small values of x can be found. When $x\gg1$ the form of Eq. (6) is governed by the behavior of the characteristic function near $u=0$. This behavior is influenced by the index γ, which stems from the Lévy-like properties rather than α which affects the clustering of steps in the walk. In Appendix B it is shown that to leading order the tails of the distribution adopt the form

$$p_\infty(x)\sim\frac{2^{2-n}\Gamma(n+\gamma)(1+\gamma/2)\sin(\pi\gamma/2)}{\Gamma(n/2)\Gamma(1/2)\Gamma((1+n+\gamma)/2)}x^{-1-\gamma}, \tag{8}$$

i.e., a power law with exactly the same index as a Lévy distribution with index γ. Thus the dimension in which the random walk takes place does not alter the form of the tails of the distribution, providing the random variables are isotropic.

New behavior emerges when $x \ll 1$. Here the form of the distribution depends upon the size of $\alpha\gamma$ compared with unity. If $\alpha\gamma > 1$ the value of $p(x)$ is finite at the origin, so the random variable possesses an inner scale. Conversely, if $\alpha\gamma < 1$, $p(x) \sim x^{\alpha\gamma-1}$ is singular (but integrable) at the origin, and clearly has a different power law from that exhibited by the tail. The special case $\alpha\gamma = 1$ divides the behavior, but also is singular at the origin when $n=1$ with $p(x) \sim -2\alpha^{\alpha}/\pi \ln(x)$ and is constant for higher dimensions. Specifically when $x \ll 1$ the limit distribution has the forms

$$p_{\infty}(x) \sim \begin{cases} \dfrac{2^{1-\alpha\gamma}\Gamma((n-\alpha\gamma)/2)\alpha^{\alpha}}{\Gamma(n/2)\Gamma(\alpha\gamma/2)} x^{\alpha\gamma-1}, & 0<\alpha\gamma<1 \\ \dfrac{\Gamma((n-1)/2)\alpha^{\alpha}}{\Gamma(n/2)\pi^{1/2}}, & \alpha\gamma=1, n>1 \\ \left(\dfrac{2^{1-n/2}}{\Gamma(n/2)}\right)^2 \dfrac{\Gamma(n/\gamma)\Gamma(\alpha-n/\gamma)\alpha^{n/\gamma}}{\gamma\Gamma(\alpha)} x^{n-1}, & \alpha\gamma>1. \end{cases}$$

Distribution (6) can be conveniently characterized using the ratio of cumulatives

$$r = \frac{\int_0^1 p_{\infty}(x)dx}{\int_1^{\infty} p_{\infty}(x)dx}.$$

If r is less than unity, the distribution has greater weight in the tail than the ‘‘front,’’ the converse being true if $r > 1$. This measure is used to chart the phase space for distribution (6) shown in Fig. 1, which is parametrized in terms of the two parameters $\alpha > 0$ and $0 \leq \gamma \leq 2$; distributions for a one-dimensional random walk are illustrated. The vertical dotted line shows the locus of $r=1$ for the Lévy distribution, which occurs at $\gamma=1$; the region to the left of this line (i.e., $\gamma < 1$) corresponds to $r<1$. The solid line is the locus of $r=1$ for distributions (6), which can be approximated by $\alpha = 3/(1-\gamma)-2$. The region to the left of this curve has $r<1$, and corresponds to where the cumulative probability of the tail is greater than that of the front, the converse being true to the right of the curve. The line for the Lévy distribution is an asymptote to this curve when $\alpha \to \infty$, and it has already been noted that Poisson number fluctuations produce a Lévy distribution in this limit. The chain curve shows the locus $\alpha\gamma=1$, above this the PDF has an inner-scale and a power-law tail, below the line it has power laws both in the tail and small values of x. Section III discusses how these distributions are modified by finite $\overline{N}$ and the manner in which the distributions converge to the limiting case.

III. DISTRIBUTIONS WITH FINITE $\overline{N}$ AND CONVERGENCE TO THE LIMIT

The analysis of Sec II is now extended to account for a finite mean number of steps in the random walk. As before, the distribution that governs N is negative binomial, with a cluster parameter α. Recall that a Lévy distribution with index γ has a power-law tail and is finite for small values of x. When N statistically identical random variables governed by these distributions are added and the mean $\overline{N} \to \infty$, singular behavior at the origin is obtained if $\alpha\gamma \leq 1$. The anticipated behavior for finite values of $\overline{N}$ is for the removal of this weak singularity at the origin, but that a vestige of the inner power-law behavior be retained, and this is indeed the case. This analysis also informs the important question of the rate of convergence to the limit distribution, which will be seen to be ultraslow.

The analysis in this section will be illustrated for a random walk in one dimension, in which case the PDF for finite $\overline{N}$, given by Eq. (4), is.

$$p_{\overline{N}}(x) = \frac{2}{\pi}\int_0^{\infty}\cos(xu)\left[1+\frac{\overline{N}}{\alpha}\left(1-\exp\left(-\frac{u^{\gamma}}{\overline{N}}\right)\right)\right]^{-\alpha} du \tag{9}$$

where scalings $u \to (\overline{N}\Lambda)^{-1/\gamma}u$, followed by $x \to (\overline{N}\Lambda)^{1/\gamma}x$ have been used. Asymptotic analysis of Eq. (9) reveals that the tail of the distribution is unchanged by the introduction of finite $\overline{N}$, so that

$$p_{\overline{N}}(x) \sim \frac{2\Gamma(1+\gamma)\sin(\pi\gamma/2)}{\pi} x^{-1-\gamma}, \quad x \gg 1$$

which is Eq. (8) with $n=1$. The behavior at other values of x can be found by rewriting the characteristic function appearing in Eq. (9) using the integral definition of the Γ function [18] in order to remove the algebraic denominator. This enables the PDF to be expressed as the double integral

$$p_{\overline{N}}(x) = \frac{2}{\pi\Gamma(\alpha)}\int_0^{\infty} dt\, t^{\alpha-1}\exp(-t) \times \exp(-\overline{N}t/\alpha)\int_0^{\infty} du \times \exp\left(\frac{\overline{N}t}{\alpha}\exp\left(-\frac{u^{\gamma}}{\overline{N}}\right)\right)\cos(ux).$$

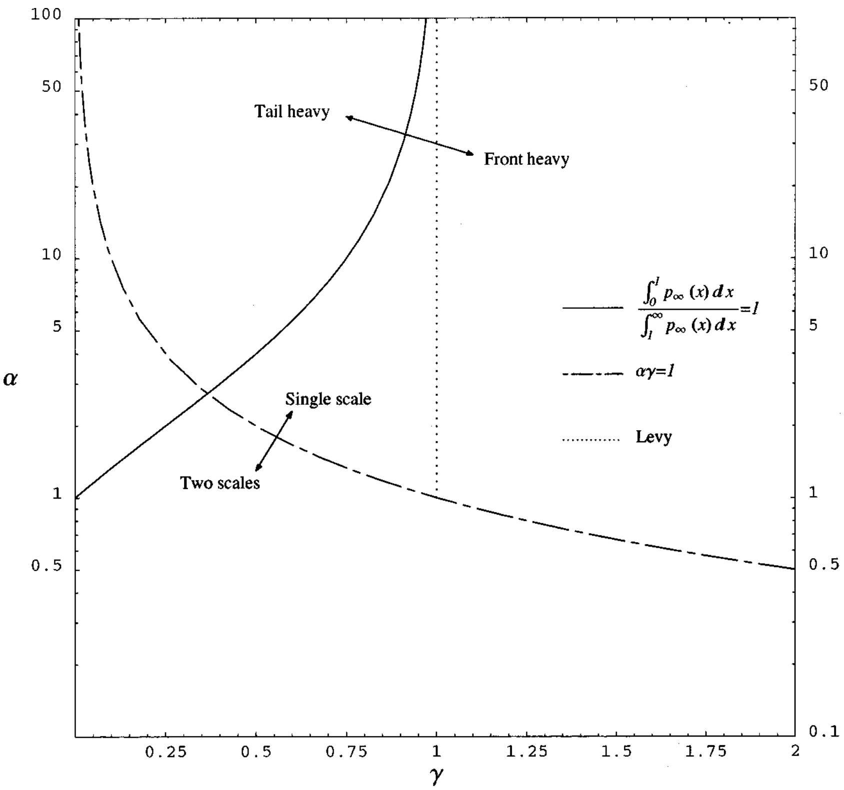

FIG. 1. A phase space of the limit distribution parametrized in terms of the Lévy index γ and the cluster parameter α. The solid curve is the locus on which the ratio of cumulants $r=1$. Distributions in regions to the left of this line have greater weight in the tail, with the front of the distribution having greater weight to the right of the curve. The vertical line is the locus for the Lévy distribution, which occurs at $\gamma=1$ and is independent of α. The chain line is the locus $\alpha\gamma=1$, above which the distributions have an inner scale and a power-law tail. Below this curve, distributions possess two power laws.

The difficulty of evaluating the ''exponential of an exponential'' can be surmounted with the aid of the approximation [33]

$$\exp\left(\frac{\bar{N}t}{\alpha}\exp\left(-\frac{u^\gamma}{\bar{N}}\right)\right)\approx 1+(\exp(\bar{N}t/\alpha)-1)\times\exp\left(-\frac{tu^\gamma}{\alpha(1-\exp(-\bar{N}t/\alpha))}\right),$$

which for large $\bar{N}$ reduces to the conventional steepest descent approximation, and for sufficiently small values of u is correct for arbitrary values of $\bar{N}$. Using this approximation evaluates the PDF as

$$p_{\bar{N}}(x)=2(1+\bar{N}/\alpha)^{-\alpha}\delta(x)+\frac{2}{\pi\Gamma(\alpha)}\int_0^\infty dt\,t^{\alpha-1}\times\exp(-t)(1-\exp(-\bar{N}t/\alpha))\int_0^\infty du\cos(ux)\times\exp\left(-\frac{tu^\gamma}{\alpha(1-\exp(-\bar{N}t/\alpha))}\right). \tag{10}$$

The δ function term represents the finite probability for there being zero steps in the random walk and hence remaining at the origin. This term is comparatively large for small $\bar{N}$, and decreases to zero as $\bar{N}\to\infty$. The integral term in the above represents the distributed part of the PDF upon which attention is now focussed. Although the integral cannot be evaluated analytically for arbitrary values of γ, its general structure can be discerned by examination of particular cases. Selecting $\gamma=1$ means the underlying distribution governing step lengths is Cauchy, and enables the integral over u to be performed:

$$p_{\bar{N}}(x)=2\delta(x)(1+\bar{N}/\alpha)^{-\alpha}+\frac{2}{\pi\Gamma(1+\alpha)}\int_0^\infty dt\,t^\alpha\exp(-t)\times\left(\left(\frac{1t}{\alpha(1-\exp(-\bar{N}t/\alpha))}\right)^2+x^2\right)^{-1}. \tag{11}$$

Appendix C 1 shows how the integral appearing in Eq. (11) can be decomposed into terms that describe the behavior at small and large values of x, viz:

$$\begin{aligned}p_{\bar{N}}(x)&\approx p_{\rm tail}(x)+p_{\rm front}(x)\\&=\frac{2}{\pi\Gamma(1+\alpha)x^2}\int_0^x dt\,t^\alpha\exp(-t)\\&\quad+\frac{2\alpha}{\pi\Gamma(\alpha)}\int_x^\infty dt\,t^{\alpha-2}(1-\exp(-\bar{N}t/\alpha))^2\exp(-t)\\&=\frac{2H(x-1/\bar{N})\gamma(1+\alpha,x)}{\pi\Gamma(1+\alpha)x^2}+\frac{2\alpha}{\pi\Gamma(\alpha)}\Bigg\{\Gamma(\alpha-1,x)\\&\quad-2\left(1+\frac{\bar{N}}{\alpha}\right)^{1-\alpha}\Gamma\left(\alpha-1,\left(1+\frac{\bar{N}}{\alpha}\right)x\right)\\&\quad+\left(1+\frac{2\bar{N}}{\alpha}\right)^{1-\alpha}\Gamma\left(\alpha-1,\left(1+\frac{2\bar{N}}{\alpha}\right)x\right)\Bigg\},\end{aligned} \tag{12}$$

where $\gamma(a,b)$ and $\Gamma(a,b)$ are the incomplete Γ functions [34], and $H(x)$ the Heaviside unit-step function [35]. Integral

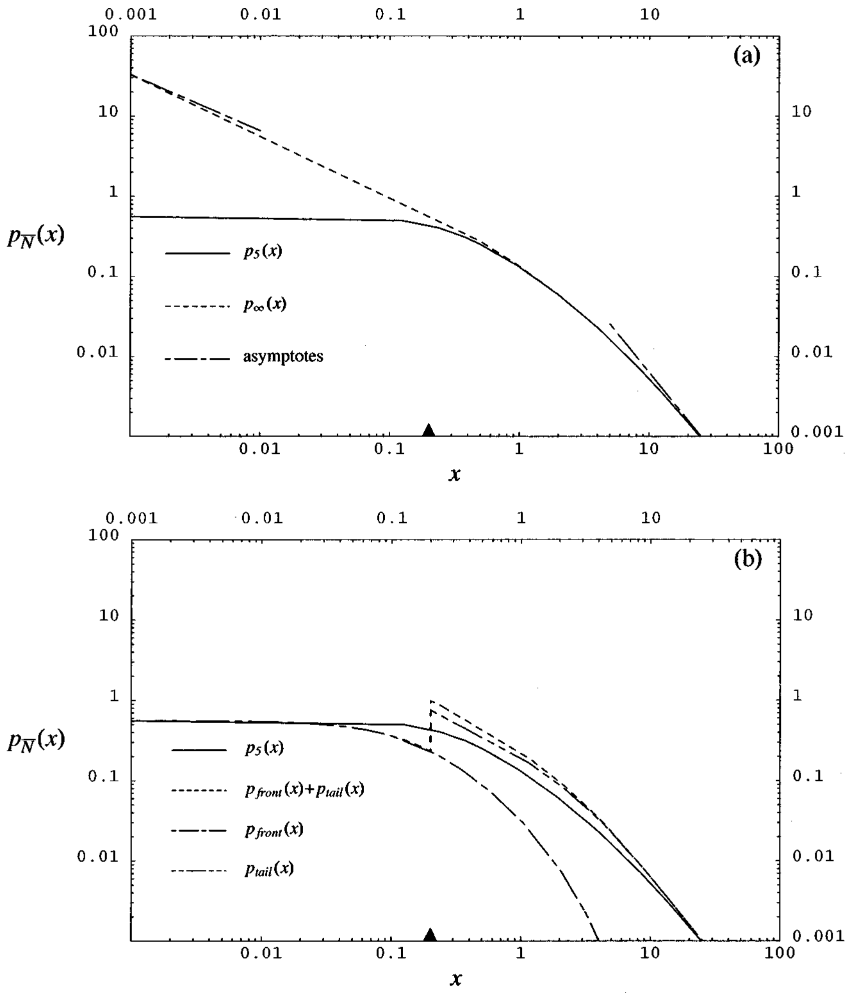

FIG. 2. Illustrating on log-log plots the PDF's that feature in the text. (a) shows the limit distribution corresponding to $\overline{N}\to\infty$ (dashed line), which is contrasted with the distribution for $\overline{N}=5$ (full line) when $\gamma=1$ and $\alpha=0.3$. Also shown are the asymptotes for small and large dimensionless values of x. The thorn on the x axis denotes the position at which the inner solution resolves the singularity that is an inherent feature of the limit distribution. (b) shows how the distribution for $\overline{N}=5$ can be constructed from the terms $p_{\mathrm{front}}(x)$ and $p_{\mathrm{tail}}(x)$, and their sum for $\gamma=1$ and $\alpha=0.3$.

(11) possesses inner, intermediate, and large-scale behavior. The inner scale is defined for $x<1/\overline{N}$, throughout which the PDF is approximately constant. For those values of x within the intermediate region $1/\overline{N}<x<1$, the PDF matches onto the small-scale asymptote of the limit distribution, so that $p(x)\sim x^{\alpha-1}$. The outer scale for $x\gg 1$ has the usual Lévy-like tail behavior with $p(x)\sim x^{-2}$.

The distribution, the various contributions for the approximation to it, and the corresponding asymptotes and scale regions are illustrated in Fig. 2. Figure 2(a) shows the limit distribution and the distribution for when $\overline{N}=5$ corresponding to the dashed and full curves, respectively: in both cases $\alpha=0.3$ and $\gamma=1$. The asymptotes derived in Sec. II are also shown. The distributions have identical tails, but exhibit different behavior at small values of x. The limit distribution is singular at the origin, but this singularity is resolved by the finite $\overline{N}$ distribution with inner scale at $\sim 1/\overline{N}$, the position of which is marked by the thorn on the x axis. Figure 2(b) compares approximation (12) with a numerical solution of Eq. (11) for when $\overline{N}=5$. The full curve gives the actual distribution, and the dot-dashed line denotes the contribution $p_{\mathrm{front}}(x)$, which provides the inner- and intermediate-scale behavior but which is negligible in the tail. The dot-dot dashed curve depicts the contribution $p_{\mathrm{tail}}(x)$, which describes the tail of the distribution, and has a cutoff at the inner scale. The dotted curve gives the sum $p_{\mathrm{tail}}(x)+p_{\mathrm{front}}(x)$. Just beyond the cutoff, both terms contribute to give an intermediate behavior that matches onto the inner power law of the limit distribution.

Another special case for which the integral over u appearing in Eq. (10) can be performed is for Gaussian distributed step lengths, i.e., for $\gamma=2$, in which case

$$p_{\overline{N}}(x)\approx\frac{\alpha^{1/2}}{\pi^{1/2}\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-3/2}\exp(-t)\times\left(1-\exp\left(-\frac{\overline{N}t}{\alpha}\right)\right)^{3/2}\times\exp\left(\frac{-\alpha(1-\exp(-\overline{N}t/\alpha))x^2}{4t}\right),$$

and, letting $\overline{N}\to\infty$, one obtains exactly the limiting K distribution

$$p_\infty(x)=\frac{\alpha^{1/2}}{\pi^{1/2}\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-3/2}\exp(-t)\exp\left(-\frac{\alpha x^2}{4t}\right)=\frac{2\alpha^{1/2}}{\pi^{1/2}\Gamma(\alpha)}\left(\frac{x}{2}\right)^{\alpha-1/2}K_{\alpha-1/2}(\alpha^{1/2}x),$$

which may be compared with Eq. (6) on setting $n=1$. Recall that the K density does not have a power-law tail, but can have an inner power-law if $\alpha<1/2$. Finite $\overline{N}$ only affects the

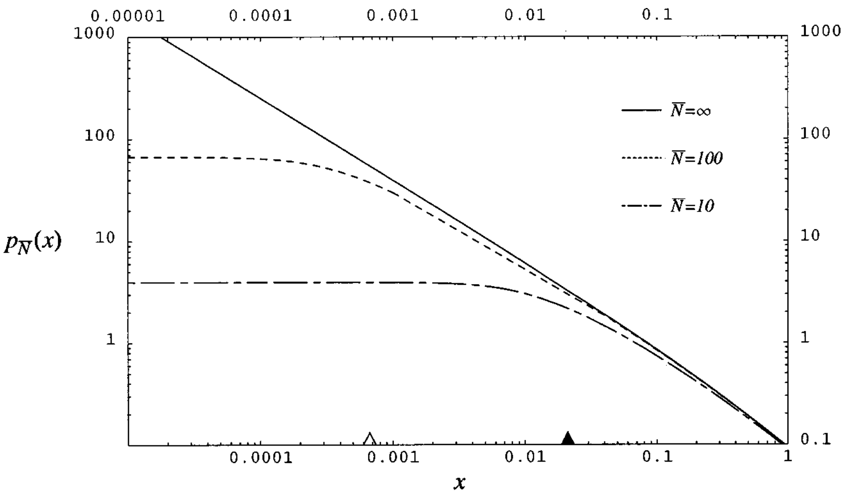

FIG. 3. Contrasting the way in which the distribution changes with $\overline{N}$ when $\gamma=\frac{2}{3}$ and $\alpha=\frac{1}{2}$. The limit distribution ($\overline{N}\rightarrow\infty$, full curve) is shown together with distributions for $\overline{N}=100$ and 10. Thorns on the x axis mark the position of the inner-scale region, the black symbol is for $\overline{N}=10$, and the open symbol for $\overline{N}=100$. All quantities are dimensionless.

inner-scale region, and once again removes the weak singularity. In Appendix C 2 the inner scale is shown to occur when $x^2<4/\overline{N}$, within which the PDF is approximately constant.

In general the inner scale occurs when

$$x\sim\gamma(\overline{N})^{-1/\gamma}, \tag{13}$$

as illustrated in Fig. 3, which contrasts the limit distribution, shown by the full curve, with the distributions for $\overline{N}=10$ and 100, shown by the dot-dashed and dotted curves, respectively. The values of the indices are $\gamma=\frac{2}{3}$ and $\alpha=\frac{1}{2}$. It can be seen that the inner scale, at locations denoted by thorns on the x axis, decreases with increasing $\overline{N}$ in accord with Eq. (13). Beyond the inner-scale region the distribution matches onto the inner power law of the limit distribution with $p(x)\sim x^{\alpha\gamma-1}$, and from there onto the tail with $p(x)\sim x^{-1-\gamma}$, as seen above.

It is clear from Figs. 2 and 3 that $p(0)$ is finite, whereas the limit distribution is singular at the origin when $0<\alpha\gamma<1$. Thus $p(0)$, given by

$$p_{\overline{N}}(0)=\frac{2\Gamma(1+1/\gamma)\alpha^{1/\gamma}}{\pi\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1-1/\gamma}\times\exp(-t)(1-\exp(-\overline{N}t/\alpha))^{1+1/\gamma}, \tag{14}$$

which is the probability of *returning* to the origin (the δ function term is the probability of *remaining* at the origin), is indicative of the rate of convergence to the limit distribution. The behavior of $p(0)$ for $\overline{N}/\alpha\ll 1$ is obtained by approximating $(1-\exp(-\overline{N}t/\alpha))$ with $\overline{N}t/\alpha$, whereupon

$$p_{\overline{N}}(0)\approx\frac{2\Gamma(1+1/\gamma)}{\pi}(\overline{N})^{1+1/\gamma}. \tag{15}$$

The behavior of $p_{\overline{N}}(0)$ in the other limit $\overline{N}/\alpha\gg 1$ is slightly more involved, and the detail for obtaining the result

$$p_{\overline{N}}(0)\approx\frac{2\gamma\Gamma(2+1/\gamma)\alpha^{\alpha}}{\pi\Gamma(\alpha)(1+\alpha)(1-\alpha\gamma)}(\overline{N})^{(1-\alpha\gamma)/\gamma} \tag{16}$$

is given in Appendix D. Note that the power-law dependence with $\overline{N}$ changes in the different regimes, and is much slower for large values of the argument. In Fig. 4 is shown a log-log plot of $p(0)$ calculated using the exact result (14), as a function of $\overline{N}$ for $\gamma=1$ (the full curve) and $\gamma=\frac{1}{2}$ (the dashed curve) for $\alpha=0.3$. Also shown are the asymptotes estimated through Eqs. (15) and (16). These curves illustrate the very slow convergence to the limit distribution. Over 100 steps in the random walk are required when $\gamma=\frac{1}{2}$ and $\alpha=0.3$, in order that $p(0)\sim 200$. This should be contrasted with the conditions for which the central limit theorem applies, when

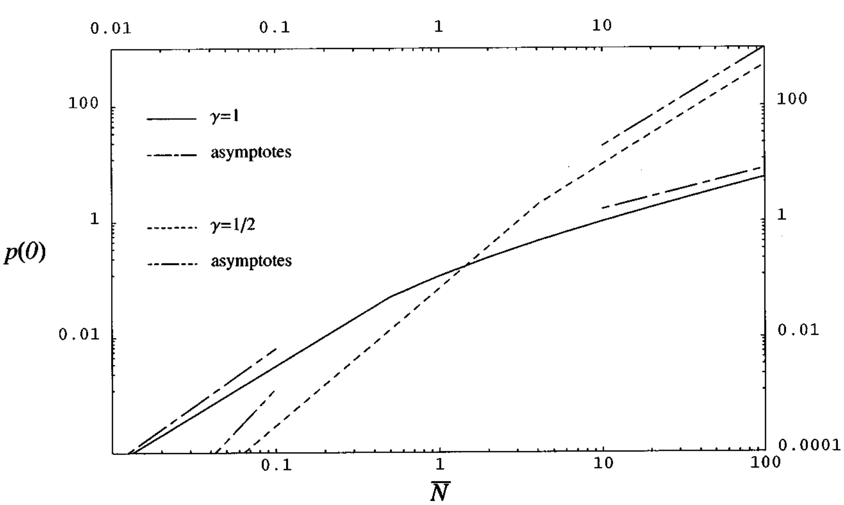

FIG. 4. Illustrating the rate of convergence to the limit distribution through the behavior of $p(0)$ when plotted as a function of $\overline{N}$. Two distributions are considered for $\gamma=1$ and $\frac{1}{2}$ and $\alpha=0.3$ in both cases. Also shown are the asymptotes derived in the text.

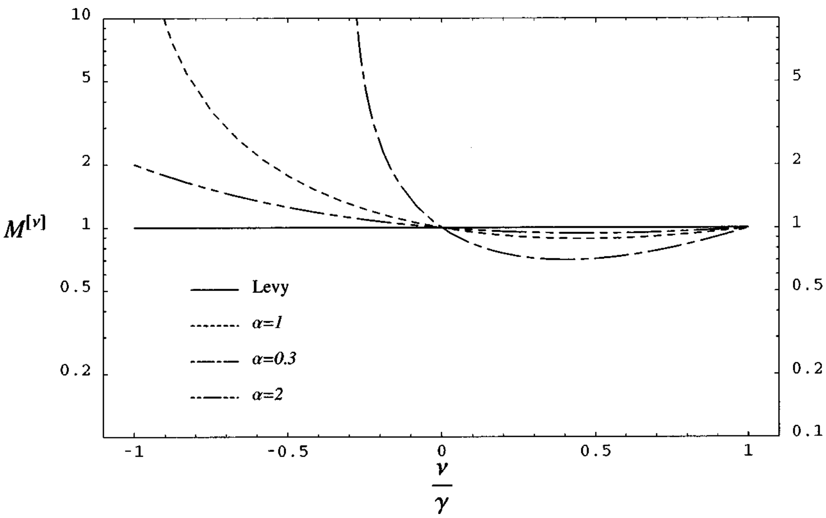

FIG. 5. Normalized fractional moments as a function of ν/γ for different values of the cluster parameter α.

convergence usually occurs for the addition of approximately ten random variables. The convergence of the truncated Lévy flight to its Gaussian limit was investigated [36], and shown to be slow. Here the convergence to the (non-Gaussian) limit distributions occurs at an even slower rate than that illustrated in Ref. [36], and has repercussions for physical systems described, or simulated, using non-Gaussian stable distributions.

IV. FRACTIONAL MOMENTS

A means of characterizing the distributions derived in this paper other than describing their asymptotic behavior is through using moments. The usual integral moments cannot be defined but fractional moments [29] can and are calculated here for distributions appropriate for a one-dimensional random walk. This section will show how these measures provide an alternative means of gauging the rate of convergence to the limit distribution and their deviation from the underlying Lévy distribution which governs the step lengths in the random walk.

The νth fractional moment is defined as

$$\langle x^{\nu}\rangle = \int_0^{\infty} x^{\nu} p_{\overline{N}}(x)\,dx\,, \tag{17}$$

and only those moments in the range $-1<\nu<\gamma<2$ exist. The νth fractional moment of the limit distribution (6) is given by

$$\langle x^{\nu}\rangle = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} x^{\nu}\cos(ux)(1+u^{\gamma}/\alpha)^{-\alpha}dx\,du,$$

which is easily determined to be

$$\langle x^{\nu}\rangle = \frac{\Gamma(-\nu/\gamma)\Gamma(\alpha+\nu/\gamma)}{\Gamma(-\nu)\Gamma(\alpha)\gamma}\sec\left(-\frac{\nu\pi}{2}\right),$$

and is valid for $\alpha\gamma>|\nu|$. The above result can be analytically continued to cover the whole parameter domain $-1<\nu<\gamma$ by rewriting the above as

$$\langle x^{\nu}\rangle = \frac{\Gamma(1-\nu/\gamma)\Gamma(1+\alpha+\nu/\gamma)(1-\nu)}{\Gamma(2-\nu)\Gamma(\alpha)(\alpha+\nu/\gamma)\alpha^{\alpha/\gamma}}\sec\left(-\frac{\nu\pi}{2}\right).$$

The equivalent fractional moment for the Lévy distribution can be determined in a similar fashion, and the normalized moment

$$M^{[\nu]} = \frac{\langle x^{\nu}\rangle}{\langle x^{\nu}\rangle_{Levy}} = \frac{\Gamma(\alpha+\nu/\gamma)}{\Gamma(\alpha)\alpha^{\alpha/\gamma}},\quad -\alpha<\nu/\gamma<1 \tag{18}$$

measures the deviation of the limit distribution from the Lévy distribution. Figure 5 shows the normalized moment (18) as a function of the parameter ν/γ for various values of α. When $\alpha=2$, the normalized moment is greater than unity for positive values of ν/γ, and this reflects the fact that the limit distribution is greater than the Lévy distribution in the tail, though it has the same asymptotic dependence on x. The limit distribution is less than the Lévy distribution near the origin, hence the negative moments of the Lévy distribution are in excess of the limit distribution. The converse to this is the case when $\alpha\leqslant 1$. The normalized moment is divergent when $\nu/\gamma=-\alpha$ if $\alpha\leqslant 1$, and this is indicative of the singular behavior of the limit distribution at the origin.

When $\overline{N}$ is finite the moment is expressible as the double integral

$$\begin{aligned}\langle x^{\nu}\rangle =\; & 2(1+\overline{N}/\alpha)^{-\alpha}\delta(x)\\ & +\frac{1}{\Gamma(-\nu)\Gamma(\alpha)}\sec\left(\frac{-\nu\pi}{2}\right)\int_0^{\infty}\int_0^{\infty} t^{\alpha-1}\\ & \times\exp(-t)(1-\exp(-\overline{N}t/\alpha))u^{-\nu-1}\\ & \times\exp\left(\frac{-tu^{\gamma}}{\alpha(1-\exp(-\overline{N}t/\alpha))}\right)du\,dt,\end{aligned}$$

where the approximation [33] used in Sec. III has been utilized once again. This expression requires numerical integration in general, but it has a structure reminiscent of the expression for $p(0)$ given by Eq. (17). This property has been alluded to elsewhere [37], although in a different context.

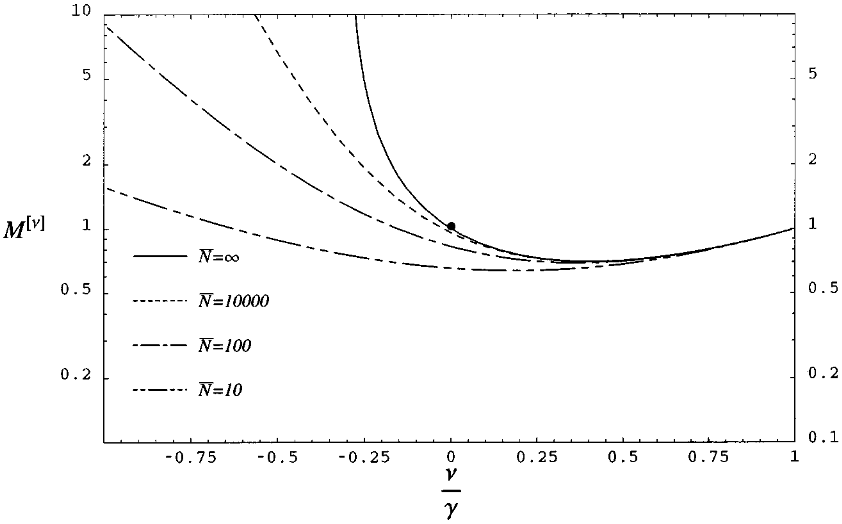

FIG. 6. Normalized fractional moments for the limit distribution and for $\overline{N}=10$, 100, and 10 000, and $\alpha=0.3$ for each case. ν denotes the order of the moment. The dot at $\nu/\gamma=0$, where $M^{[\nu]}=1$ denotes the discontinuity in the finite $\overline{N}$ distributions due to the finite probability of remaining at the origin.

For the purposes of illustration the special case when $\gamma=1$ is considered which enables the integration over u to be performed to yield

$$\langle x^{\nu}\rangle=2(1+\overline{N}/\alpha)^{-\alpha}\delta(x)+\frac{\gamma(1-\nu)}{\pi\Gamma(\alpha)\Gamma(2-\nu)}\times\Gamma\left(1-\frac{\nu}{\gamma}\right)\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{1+\nu}{2}\right)\int_0^{\infty}t^{\alpha-1-\nu/\gamma}\times\exp(-t)(1-\exp(-\overline{N}t/\alpha))^{1-\nu/\gamma}dt,$$

and this exists over the range $-1<\nu<1$. The normalized moments for ν in this range are shown in Fig. 6 when $\alpha=0.3$ for values of $\overline{N}=10$, 100, and 10 000, together with the moments for the limit distribution. The moments for the limit distribution diverge when $\nu>\gamma=1$, which is caused by the power-law tail. The moments also diverge when $\nu/\gamma+\alpha<0$, corresponding to $\nu=-0.3$, and this behavior is caused by an inner power law occurring at small values of x. The moment is necessarily unity when $\nu=0$. The moments for finite $\overline{N}$ also diverge when $\nu=1$, because the distributions have exactly the same behavior in the tail as the limit distribution. The moment at $\nu=0$ is discontinuous, but again is necessarily equal to unity; a dot marking the values the finite $\overline{N}$ moments adopt is shown on the figure. The distance between the continuous curves and the dot measures the contribution of the δ-function which represents the finite probability of remaining at the origin. When $\overline{N}$ is large, this contribution is necessarily small as can be seen from the figure. The finite $\overline{N}$ distributions are not singular at the origin, and so moments of these distributions are finite up to values of $\nu=-1$. Figure 6 again illustrates the ultraslow rate of convergence to the limit distribution. Although the fractional moments coincide for sufficiently large and positive values of ν, there are marked differences for negative values of ν, even when $\overline{N}$ is as large as 10 000.

V. DISCUSSION

In earlier sections various generalizations of previous work on stable distributions have been described. As a mathematical exercise this has provided a number of analytical challenges, and has enabled parameter regimes under which probability densities may exhibit power-law behavior to be identified. The principal innovation of this work has been to introduce step number fluctuations associated with a particular kind of clustering process into the random walk problem. This approach is motivated through recognizing that step number fluctuations provide a means for introducing correlations into the formulation, albeit in a phenomenological fashion. Analysis has shown that this leads to the possibility that the probability density of the resultant of the random walk can exhibit two different power-law regimes. This kind of behavior was observed [10] in numerical experiments based on a simple cellular automaton model which was used to reproduce the transport properties of grains in a rice pile [9]. These authors found that the distribution of trapping times in the pile could be represented by a power law of approximately -0.97 extending over two decades at small trapping times, and another near -2.2 extending over more than three decades at large trapping times. While recognizing the physical origin of the second regime, they failed to comment on the first, and it is indeed not obvious why this regime is predicted by their model. The relationship between the different power laws appearing in the limiting distribution derived in Sec. II is

$$s_f+\alpha s_t+1+\alpha=0,$$

where s_t and s_f denote the power-indices in the tail and at small scales, respectively. Using the values for s_t and s_f quoted in Ref. [10] predicts $\alpha=\frac{1}{40}$. Figure 7(a) compares the trapping time distribution that features in Fig. 1 of Ref. [10] with the limit distribution of Sec. II. The asymptotes quoted in Ref. [10] are shown by the chain lines; these are scaled according to $p(x)\rightarrow 2p(x/10)$, and depicted by the full curve. The limit distribution with the predicted value of α is shown by the dotted curve. Figure 7(b) performs a similar comparison with the distribution of flight lengths that fea-

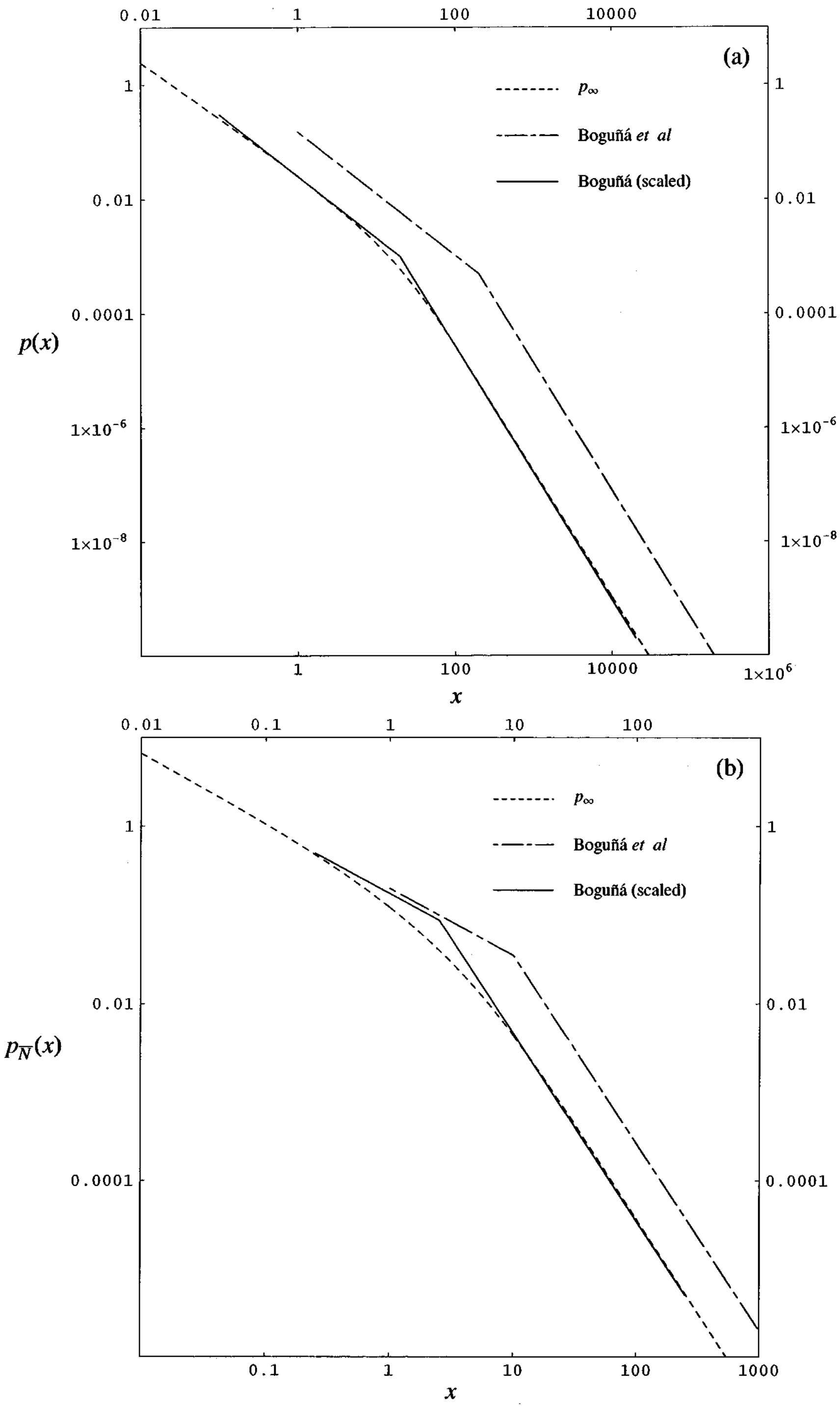

FIG. 7. Comparisons of the distributions obtained from the cellular automata results of Ref. [10] for trapping times (a) and flight lengths (b) with the limit distribution with parameters obtained from the data. Units are arbitrary.

tures in Fig. 2 of Ref. [10]. There the tail has a power law of approximately -2.13 which extends over two decades, while at small scales the power law is -0.76 and extends over a single decade. Again, the chain line shows the asymptotes, the full curve corresponds to the scaling $p(x)\rightarrow 2.5p(x/4)$, and the dotted line is the limit distribution with α predicted to be ~ 0.21. In both cases the limit distribution compares favorably with the scaled cellular automata data.

Although the number fluctuation process proposed in this paper appears at first sight to be somewhat *ad hoc*, the underlying model possesses a number of features which one might expect to find associated with natural phenomena. Its significance in the present context can be better appreciated through a brief review of previous work on the effect of number fluctuations on random walks.

Random walks in two dimensions are often used to model the scattering of electromagnetic waves by random media such as rough surfaces. In this approach the scattered complex electromagnetic field amplitude is represented as a sum of randomly phased contributions from different scattering centers. Here there is generally no relationship between the number of scattering centers and the amplitude contributed by each scatterer, corresponding to the number of steps in the walk and the length of each step, respectively. In a given illuminated area or volume of the scattering medium the number of scattering centers may be fixed but, if the medium is evolving in time or moving relative to the illuminating beam, then the number of scatterers will change. For example, in the case of small particles in suspension which are moving in and out of an illuminated volume this process will be governed by a Poisson distribution (the probability aftereffect). On the other hand, a scattering system like the sea surface contains inhomogeneities of many sizes. One is readily convinced by visual observation that large waves, current, wind, and surface contamination tend to modulate the structure of the smaller waves. The local behavior of descriptors such as surface height and slope is *intermittent* or *inhomogeneous* in such circumstances: a common feature of natural multiscale phenomena such as turbulence but not necessarily indicating the existence of a fractal cascade of sizes.

When microwaves are scattered from the sea surface, inhomogeneity is manifest as a modulation of the density of scatterers that often appear to be clustered near wave crests. The negative binomial cluster model has been used to characterize this effect in the development of a simple non-Gaussian model for the amplitude statistics of the scattered electromagnetic field based on a random walk approach. For this system, the individual mean square scattering amplitudes (step lengths) represent the cross sections of the scatterers and are finite. In the high-density limit, K distributions are predicted that are *independent* of the statistical properties of the individual steps. The case $\gamma=2$ of the present paper falls into this category of problem. The properties of the number fluctuation model then entirely determine the form of the limit distribution. A necessary condition for the limit distribution to be non-Gaussian is that the relative variance of the number fluctuations should not vanish as the average number becomes large. This condition is not satisfied by the Poisson distribution, for example. Note that K distributions admit a compound representation in which the mean square amplitude of a Gaussian distribution is modulated by a Gamma distribution [38]. In the electromagnetic scattering context this means that K distributions can be interpreted as being a random interference pattern with a spatially or temporally varying local brightness. Although not of the stable class, K distributions are infinitely divisible, which means that the coherent addition of K-distributed variables is governed by a distribution belonging to the same class but with a reduced variance. This useful property is inherited from the parent gamma density.

The earlier results described above may be contrasted with the calculations of Sec. II. These show that when the step lengths of a random walk are Lévy distributed with $\gamma<2$ (i.e., with infinite variance), the tail of the limit distribution of the resultant amplitude *preserves* the power-law behavior of the distribution characterizing the length of individual steps. This result appears to be independent of the choice of number fluctuation distribution. In the high density limit a compound representation in the sense described above is clearly not possible. However, it is possible to interpret the continuum limit as representing variations in the anomalous diffusion coefficient Λ appearing in Eqs. (3)–(5). When the mean number of steps is very large, number fluctuations only affect the probability density provided that their relative variance remains nonzero, and then mainly in the regime of small excursions. A strong effect, manifested by the appearance of a new power-law regime, is dependent on a trade-off between the behavior of the variations in step number and step length. In the case of the negative binomial model a second power law is predicted for all allowed values of γ only when $\alpha<\frac{1}{2}$. It is perhaps significant that the continuum analog of the number distribution is singular at the origin when this criterion is satisfied.

The choice of the negative binomial number fluctuation distribution has several advantages. Apart from being a model whose variance remains finite and nonzero in the high density limit, it is the equilibrium distribution of a fundamental and tractable stochastic population model which is well documented in the literature [31]. This has enabled some progress to be made in developing the higher order joint statistical properties of K-distributed noise [28]. From a phenomenological viewpoint, a naive description of three-dimensional turbulent flow, namely, the spontaneous nucleation or ''immigration'' of large eddies which ''give birth'' to smaller eddies which eventually ''die'' due to viscous dissipation, is analogous to the birth-death-immigration process. In the electromagnetic scattering problem the size of eddies is subordinated to fluctuations in their number. On the other hand, in the case of a constant speed random walk of given duration whose changes of direction are governed by ''death'' events in a birth-death-immigration process, the number and length of steps are correlated.

It is interesting that the distribution of inter-event times (i.e., step lengths) for the birth-death-immigration process exhibits a range of power-law behavior. An exact result can be derived from the generating function of the number of events in a finite time interval of duration T [39]:

$$Q(s;T)=\sum_{N=0}^{\infty}(1-s)^N p(N;T)$$

$$=\frac{\exp(\alpha\Gamma T)}{[\cosh yT+(y/2\Gamma+\Gamma/2y)\sinh yT]^{\alpha}},$$

where $1/\Gamma$ is the characteristic bunching time of the number fluctuations, $y^2=\Gamma^2+2\bar{R}\Gamma(1-s)/\alpha$, and $\bar{R}=\bar{N}/T$ is the average event rate. This result reduces to the generating function for a negative binomial distribution in the limit $\Gamma T\ll 1$ when there is no time averaging of the number fluctuations. Using the generating function the distribution of intervals between events can be determined. When t is much less than the characteristic time of the number fluctuations, this is [39]

$$p_1(t)\approx\frac{\bar{R}(1+1/\alpha)}{(1+\bar{R}t/\alpha)^{\alpha+2}}.$$

If the events are sufficiently frequent then an inverse power law in t is obtained. This type of behavior has been heuristically invoked by several groups recently in the context of Lévy flights and SOC [10, 40]. Note that the negative binomial model exhibits inner and outer scales, where the average inter event period is $\alpha/\bar{R}$ and the correlation time is $1/\Gamma$, in addition to a power-law region.

VI. SUMMARY AND CONCLUSIONS

This paper has examined random walks where the distribution of individual steps comprising the walk is governed by stable distributions but where the number of steps in the walk can fluctuate and be subject to clustering. Such an innovation to the Lévy-flight literature is informed by its success in the context of optical scattering from random media where step fluctuations model correlated phenomena, such as small-scale modulation by larger scale structures. The model chosen for the step number fluctuations is the negative-binomial distribution, which is the steady state of a birth-death-immigration process. This has Poisson and Bose-Einstein (geometric) number fluctuations as special cases, and in general the Fano factor, $Var(N)/\overline{N}=\overline{N}/\alpha\geqslant 1$, usually for the negative binomial class. Distributions for the distance of the resultant of the random walk from the origin are obtained, both in the limit when the average number of steps is infinite, and also when it is finite. The tails of the distributions exhibit a power law behavior that has the same characteristics as the underlying Lévy distribution. This result is true irrespective of the number of steps taken in the random walk. Clustering can lead to the resultant having qualitatively different behavior from the underlying Lévy properties that describe the individual steps. When the product formed by the cluster parameter and the index of the Lévy distribution is less than or equal to unity, a different power law at small scales is introduced, whereas, if this product is greater than unity, the density is finite at the origin. When the cluster parameter is infinite, the number of steps is uncorrelated and described by Poisson fluctuations, and in this case the limit distribution is exactly a Lévy distribution, so that the Lévy-Gnedenko generalization of the central limit theorem applies. When the individual step lengths are Gaussian, the K distribution is obtained.

Walks comprising a finite number of steps always have an inner scale, but a vestige of the inner power-law behavior is retained in an intermediate region if $\alpha\gamma\leqslant 1$. Beyond this region the distribution matches onto the Lévy-like tail. The size of the inner scale depends on an inverse power of the number of steps. The other noteworthy feature of these distributions is the δ function contribution at the origin. This represents the finite probability for their being no steps in the random walk, and thereby remaining at the origin.

The convergence to the limit distribution is ultraslow. A random walk comprising, on average, 100 steps was shown to differ substantially from the limiting form, these differences occurring for small values of x. Once again the form of the tail for the distribution is insensitive to the number of steps.

The two parameter limit distributions derived in this paper bear a striking resemblance to distributions obtained from cellular automata computer simulations that have been purported to explain SOC behavior in experimental rice piles. Indeed, distributions obtained from the experimental rice pile of Ref. [9] show some semblance of a dual power-law behavior, though the authors did not comment upon this. The precise reason for this apparent similarity is unknown at present, chiefly because the correspondence between the random variable appearing in the respective works is unclear. Seeking the reason for any such correspondence requires further detailed investigation and may, as a result, also explain the power-law tails that are a feature of SOC systems in addition to those occurring at small scales. In addition to this program of work, the present approach can be generalized further by considering the properties of anisotropic Lévy random walks with fluctuating steps. Such a generalization may be of value to analyzing anomalous transport in systems with broken spatial symmetry, such as confined magnetized plasma, among others.

ACKNOWLEDGMENT

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APPENDIX A

This appendix generalizes the results of Ref. [16] by formulating the Tsallis entropy for an n-dimensional random variable which is then varied subject to constraining the renormalized rth moment in order to find the distribution function for a single step in an n-dimensional random walk. The random variable $\mathbf{x}$ is an isotropic n vector of distance $x=|\mathbf{x}|$ from the origin and with the remaining $n-1$ polar components distributed uniformly over the surface of a hypersphere. The Tsallis entropy is defined as [19]

$$S_q[p]=\frac{1}{(q-1)}\left(1-\frac{1}{\sigma^{r/2}}\int(\sigma^{r/2}p(\mathbf{x}))^q dx\right),\qquad\text{(A1)}$$

which is parametrized by the real number q. When $q<1$, the effect of Eq. (A1) is to apply weight to those events for which $x\ll 1$, while $q>1$ biases the events $x\gg 1$. When $q=1$, Eq. (A1) reduces to the familiar integral of $-p\ln p$. The stationary values of this functional are to be found for when the PDF $p(\mathbf{x})$ has unit normalization, and the rth moment, as defined through its q expectation value, is constrained. The rth moment is defined as

$$n\sigma^r=\langle\mathbf{x}^r\rangle_q=\int x^r(\sigma^{r/2}p(\mathbf{x}))^q d\mathbf{x}$$

$$=\Omega_n\int_0^\infty x^{r+n-1}(\sigma^{r/2}p(x))^q dx\qquad\text{(A2)}$$

with $\Omega_n=2\pi^{n/2}/\Gamma(n/2)$ being the surface area of an n-dimensional unit hypersphere, and $\Gamma(x)$ the Γ function [18]. Here $r>0$ is any real number and the existence of moments is ensured for a range of values of q through using the q expectation [Eq. (A2)], which effectively renormalizes the divergent moments of the Lévy distribution. Performing a variation on Eq. (A1) with respect to p determines the most likely distribution contingent upon satisfying the applied constraints, that is,

$$\delta\left[S_q[p]+\alpha\left(1-\int p(\mathbf{x})d\mathbf{x}\right)+\beta(1-\langle\mathbf{x}^r\rangle_q)\right]=0,$$

where α and β are Lagrange multipliers whose values provide the normalization and scaling of x in the distribution, respectively. The PDF's that are obtained are a generalization of the Cauchy distribution [30], and have qualitatively different behaviors depending upon whether q is greater or less than unity, in accord with remarks made concerning the biasing effect of the Tsallis entropy. Specifically,

$$p(x)=\begin{cases} A_q(n,r)x^{n-1}(1-x^r)^{1/(1-q)} & \text{for } -\infty<q<1, \quad 0\leqslant x\leqslant 1 \\ \dfrac{r^{1-n/r}x^{n-1}}{\Omega_n\Gamma(n/r)\sigma^n}\exp(-x^r/r\sigma^r), & q=1, \quad 0\leqslant x<\infty \\ B_q(n,r)x^{n-1}(1+x^r)^{-1/(q-1)} & \text{for } 1<q<1+r/n, \quad 0\leqslant x<\infty, \end{cases} \tag{A3}$$

with the normalizing functions defined by

$$A_q(n,r)=\frac{r\Gamma\left(1+\dfrac{1}{1-q}+\dfrac{n}{r}\right)}{\Omega_n\Gamma\left(1+\dfrac{1}{1-q}\right)\Gamma\left(\dfrac{n}{r}\right)x_0^n}, \quad B_q(n,r)=\frac{r\Gamma\left(\dfrac{1}{q-1}\right)}{\Omega_n\Gamma\left(\dfrac{1}{q\text{-}1}-\dfrac{n}{r}\right)\Gamma\left(\dfrac{n}{r}\right)x_0^n}.$$

The scale parameter x_0 is related to the Lagrange multiplier through $x_0^r=1/|1-q|\beta$, which can be determined by substituting Eq. (A3) into Eq. (A2):

$$x_0^{n(1-q)+r}=\begin{cases} \dfrac{r\sigma^{(3-q)r/2}}{|1-q|}\left(\dfrac{\Omega_n\Gamma(1+1/(1-q))\Gamma(n/r)}{r\Gamma(1+1/(1-q)+n/r)}\right)^{q-1} & \text{for } -\infty<q<1, \\ \dfrac{r\sigma^{(3-q)r/2}}{|1-q|}\left(\dfrac{\Omega_n\Gamma(1/(q-1)-n/r)\Gamma(n/r)}{r\Gamma(1/(q-1))}\right)^{q-1} & \text{for } 1<q<1+r/n. \end{cases} \tag{A4}$$

For values of q outside the range specified in Eq. (A3), the constraint on the q expectation of the r-th moment cannot be satisfied. These results generalize those given in Ref. [16] to n dimensions and to where a moment of arbitrary order r is constrained to be finite. Those moments of order $m<r$ will be similarly constrained.

The characteristic function $C^q(u)$ of the single step distributions (A3) enable the resultant of the addition of N such statistically identical random variables to be determined. Two cases require consideration depending upon whether q is less than or greater than unity.

Case (a) $-\infty<q\leqslant 1$. In this case it is convenient to write the characteristic function as

$$C^q(u)=\frac{A_q(n,r)(2\pi)^{n/2}x_o^{1+n/2}}{u^{n/2-1}}\times\int_0^1 x^{n/2}J_{n/2-1}(xx_ou)(1-x^r)^{1/(1-q)}dx.$$

Expanding the Bessel function for small values of u gives

$$\begin{aligned} C^q(u)&=\frac{A_q(n,r)(2\pi)^{n/2}x_o^n}{2^{n/2-1}\Gamma(n/2)}\int_0^1 x^{n-1}(1-x^r)^{1/(1-q)} \\ &\quad\times\left(1-\frac{\Gamma(n/2)}{\Gamma(1+n/2)}\left(\frac{xx_ou}{2}\right)^2+\cdots\right)dx \\ &=1-\frac{\Gamma(1+1/(1-q)+n/r)\Gamma((2+n)/r)x_o^2}{2n\Gamma(n/r)\Gamma(1+1/(1-q)+(n+2)/r)}u^2+\cdots \\ &\approx\exp(-\Lambda_q(n,r)u^2) \end{aligned}$$

to accuracy $O(u^2)$. Hence the characteristic function is asymptotic to that of a Gaussian, and the addition of N statistically identical random variables will yield a characteristic function which is also Gaussian but with scale $N\Lambda_q(n,r)$.

Case (b) $1<q\leqslant 1+r/n$. This instance requires evaluation of

$$C^q(u)=\frac{B_q(n,r)(2\pi)^{n/2}}{u^{n/2-1}}\times\int_0^\infty x^{n/2}J_{n/2-1}(x)[1+(x/(x_ou))^r]^{-1/(q-1)}dx,$$

and this integral has different asymptotic behavior depending upon the size of x/x_ou compared with unity. When $x/x_ou\ll 1$, the change of variable $x\to xu$ enables the Bessel function to be expanded in powers of u to yield

$$\begin{aligned} C^q(u)&=\frac{B_q(n,r)(2\pi)^{n/2}}{2^{n/2-1}\Gamma(n/2)}\int_0^\infty x^{n-1}(1+(x/x_o)^r)^{-1/(q-1)} \\ &\quad\times\left(1-\frac{\Gamma(n/2)}{\Gamma(1+n/2)}\left(\frac{xu}{2}\right)^2+\cdots\right)dx \\ &=1-\frac{\Gamma(1/(q-1)-(n+2)/r)\Gamma((2+n)/r)x_o^2}{2n\Gamma(n/r)\Gamma(1/(q-1)-n/r)}u^2 \\ &\quad+\cdots \qquad \text{for } q<1+r/(n+2) \\ &\approx\exp(-\Lambda_q(n,r)u^2) \end{aligned}$$

to $O(u^2)$. Once again the behavior is asymptotic to a Gaussian random variable for values of q in the range stated.

When $x/x_o u \gg 1$ the asymptotic behavior different and to leading order is:

$$C^q(u)=1+\frac{B_q(n,r)(2\pi)^{n/2}x_o^{r/q-1}}{u^{n-r/(q-1)}}\int_0^\infty x^{n/2-r/(q-1)}J_{n/2-1}(x)dx+\cdots$$

$$=1-\frac{\Gamma\left(\frac{1}{(q-1)}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(1+\frac{1}{2}(n-r/(q-1))\right)}{\Gamma\left(\frac{n}{r}\right)\Gamma\left(\frac{r}{(2(q-1))}\right)\Gamma\left(1+\frac{1}{q-1}-\frac{n}{r}\right)}\left(\frac{ux_o}{2}\right)^{-n+r/(q-1)}+\cdots \qquad \text{for } 1+r/(n+2)<q<1+n/r$$

$$\approx \exp(-\Lambda_q(n,r)u^\gamma) \quad \text{where} \quad \gamma=-n+r/(q-1)<2$$

which is the characteristic function of a symmetric Lévy distribution. Thus when $-\infty<q<1+r/(n+2)$, $C^q(u)\sim\exp(-\Lambda_q u^2)$ for small values of u, and the statistical properties of a random walk are asymptotic to a Gaussian random process that leads to normal Brownian diffusion. When q falls in the range $1+r/(n+2)<q<1+r/n$, $C^q(u)\sim\exp(-\Lambda_q u^\gamma)$, with $\gamma=-n+r/(q-1)$, where $0<\gamma<2$ which defines a Lévy distribution. In this case the random walk is asymptotic to ''fractional'' Brownian motion leading to anomalous diffusion. For values of $q>1+r/n$, the renormalized rth moment is not defined. The scaling function Λ_q is related to the diffusion coefficient through evaluation of the appropriate moment given by Eq. (A2). In each of the regimes it is given by

$$\Lambda_q=\begin{cases}\dfrac{\Gamma\left(1+\frac{1}{1-q}+\frac{n}{r}\right)\Gamma\left(\frac{2+n}{r}\right)}{2n\Gamma\left(\frac{n}{r}\right)\Gamma\left(1+\frac{1}{1-q}+\frac{n+2}{r}\right)}x_o^2, & -\infty<q<1\\[2ex] \dfrac{\Gamma\left(\frac{1}{q-1}-\frac{n+2}{r}\right)\Gamma\left(\frac{2+n}{r}\right)}{2n\Gamma\left(\frac{n}{r}\right)\Gamma\left(\frac{1}{q-1}-\frac{n}{r}\right)}x_o^2, & 1<q<1+r/(n+2)\\[2ex] \dfrac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(1+\frac{r}{2}\left(\frac{n}{r}-\frac{1}{q-1}\right)\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{r}\right)\Gamma\left(\frac{r}{2(q-1)}\right)\Gamma\left(1+\frac{1}{q-1}-\frac{n}{r}\right)}\left(\frac{x_o}{2}\right)^\gamma, & 1+r/(n+2)<q<1+r/n,\end{cases}$$

with x_o given by Eqs. (A5). The function Λ_q is continuous as q passes through 1, but exhibits a discontinuity as the regime defining the Lévy-like behavior is broached. To the left of the discontinuity the diffusion is Gaussian and to the right it is anomalous. The coefficient is not defined for values of $q>1+r/n$, since the q expectation of the rth moment is divergent in this regime.

APPENDIX B

This appendix outlines the analysis for obtaining the asymptotic form of the distributions. First the form for $x\gg1$ is obtained. Distribution (6) can be recognized as the Hankel transform of the function

$$u^{(n-1)/2}(1+u^\gamma/\alpha)^{-\alpha},$$

which may be expanded near the origin using

$$u^{(n-1)/2}\exp(\ln(1+u^\gamma/\alpha)^{-\alpha})$$

$$=u^{(n-1)/2}\left(1-u^\gamma+\frac{(1+\alpha)}{2}u^{2\gamma}-\frac{1}{3}\left(1+\frac{3}{2}\alpha-\frac{1}{2}\alpha^2\right)u^{3\gamma}+\cdots\right)$$

$$=\sum_{m=0}^\infty k_m u^{a_m},$$

defining the coefficients k_m and indices a_m, the first four terms of which are given in Table I.

Substituting this expansion into Eq. (6) and writing $\nu=xu$ enables the PDF to be written as the expansion

$$p(x)\sim\frac{2^{1-n/2}}{\Gamma(n/2)}x^{(n-1)/2}\sum_{m=0}^\infty\frac{k_m}{x^{a_m+1}}\int_0^\infty \nu^{a_m+1/2}J_{n/2-1}(\nu)d\nu.$$

The first nonzero contribution occurs for $m=1$, which yields Expression (10) given in the text.

TABLE I. Coefficients of the asymptotic expansion for the tail of the distributions.

m	k_m	a_m
0	1	$\frac{(n-1)}{2}$
1	-1	$\frac{(n-1)}{2}+\gamma$
2	$\frac{1}{2}(1+\alpha)$	$\frac{(n-1)}{2}+2\gamma$
3	$\frac{-1}{3}\left(1+\frac{3\alpha}{2}-\frac{\alpha^2}{2}\right)$	$\frac{(n-1)}{2}+3\gamma$

APPENDIX C

The following appendixes show how the inner scale associated with the PDF's describing random walks with a finite average number of steps arise.

1. Cauchy distributed steps

Analysis of the integral in Eq. (11),

$$\frac{2}{\pi\Gamma(1+\alpha)}\int_0^\infty dt\, t^\alpha \times\exp(-t)\left(\left(\frac{t}{\alpha(1-\exp(-\bar{N}t/\alpha))}\right)^2+x^2\right)^{-1},$$

proceeds by dividing the range of integration into regions where one of the terms comprising the denominator dominates the other. The first region corresponds to values of t that are sufficiently small for the denominator to be approximated by x^2, i.e., when

$$t/[\alpha(1-\exp(-\bar{N}t/\alpha))]<x, \tag{C1}$$

in which case:

$$p_{\text{tail}}(x)\approx\frac{2}{\pi\Gamma(1+\alpha)x^2}\int_0^x dt\, t^\alpha\exp(-t).$$

Note that for large values of x, this expression $\sim 2/\pi x^2$, which is precisely the form for the tail of the distribution according to Eq. (8). The apparent divergent behavior as $x\to 0$ is prevented through condition (C1) which imposes a smallest allowable value of $x\sim 1/\bar{N}$ that defines an inner scale within which the approximation $p_{\text{tail}}(x)$ is invalid. Hence

$$p_{\text{tail}}(x)\approx\frac{2H(x-1/\bar{N})\gamma(1+\alpha,x)}{\pi\Gamma(1+\alpha)x^2} \tag{C2}$$

where $\gamma(a,b)$ is the incomplete gamma function [34], and $H(x)$ is the Heaviside unit-step function [35].

The second region corresponds to values of t for which the first term in the denominator of the integrand of Eq. (11) exceeds the second. This occurs when $x<t<\infty$, where x is chosen so that $t/(1-\exp(-\bar{N}t/\alpha))>x$, and in this regime a value of t can be found for *any* value of x. Thus

$$\begin{aligned}p_{\text{front}}(x)&\approx\frac{2\alpha^2}{\pi\Gamma(1+\alpha)}\int_x^\infty dt\, t^{\alpha-2}\times\exp(-t)(1-\exp(-\bar{N}t/\alpha))^2\\&=\frac{2\alpha}{\pi\Gamma(\alpha)}\left\{\Gamma(\alpha-1,x)-2\left(1+\frac{\bar{N}}{\alpha}\right)^{1-\alpha}\times\Gamma\left(\alpha-1,\left(1+\frac{\bar{N}}{\alpha}\right)x\right)+\left(1+\frac{2\bar{N}}{\alpha}\right)^{1-\alpha}\times\Gamma\left(\alpha-1,\left(1+\frac{2\bar{N}}{\alpha}\right)x\right)\right\},\end{aligned} \tag{C3}$$

where $\Gamma(a,b)$ is the complementary incomplete gamma function [34]. Expression (C3) is finite at $x\approx 0$ and provides an excellent approximation to $p(x)$ within the inner-scale region, as may be seen from Fig. 2(b). Equations (C2) and (C3) together show that the PDF is approximately constant and given by Eq. (C3) for $0<x<1/\bar{N}$. For $x\sim 1/\bar{N}$ both Eqs. (C2) and (C3) contribute to the value of the density. Indeed, expanding the incomplete gamma functions and evaluating the resultant for $x\sim 1/\bar{N}$ shows that

$$p_{\bar{N}}(x)\approx p_{\bar{N}}(0)+\frac{2}{\pi\Gamma(2+\alpha)}x^{\alpha-1}+\cdots \qquad \text{for } x>1/\bar{N},$$

indicating that beyond the inner scale region, the PDF matches the power law associated with the limit distribution. For larger values of x, contribution (C3) which decreases monotonically, is negligible and the PDF can be approximated by Eq. (C2)

2. Gaussian distributed steps

The analysis of Eq. (13), which occurs for the case $\gamma=2$, has distributed part of the PDF.

$$p_{\bar{N}}(x)\approx\frac{\alpha^{1/2}}{\pi^{1/2}\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-3/2}\times\exp(-t)(1-\exp(-\bar{N}t/\alpha))^{3/2}\times\exp\left(\frac{-\alpha(1-\exp(-\bar{N}t/\alpha))x^2}{4t}\right),$$

and analysis proceeds in a similar fashion to that for Eq. (11), except in this instance there are no separate regimes of behavior. This is because the tail of the distribution is not a power law; rather it has an exponential behavior. The integral will be approximately constant if the argument of the last exponential appearing in the integrand is small. This occurs when

$$(1-\exp(-\bar{N}t/\alpha))\approx\bar{N}t/\alpha,$$

and so the inner scale within which the PDF is constant occurs for values of x satisfying $x^2 \ll 4/\overline{N}$. The distribution is asymptotic to the tail of the distribution when $x^2 \gg 4/\overline{N}$.

APPENDIX D

The asymptotic behavior that is used to characterize $p_{\overline{N}}(0)$ in the limit $\overline{N}/\alpha \gg 1$ is calculated by first writing $\nu = \overline{N}t/\alpha$ to give

$$p_{\overline{N}}(0) \approx \frac{2\Gamma(1+1/\gamma)\alpha^{-\alpha}}{\pi\Gamma(\alpha)}(\overline{N})^{1/\gamma-\alpha} \times \int_0^\infty \nu^{\alpha-1-1/\gamma}(1-\exp(-\nu))^{1+1/\gamma} \times \exp\left(-\frac{\alpha\nu}{\overline{N}}\right)d\nu.$$

The integral is less than

$$\int_0^\infty \nu^{\alpha-1-1/\gamma}(1-\exp(-\nu))^{1+1/\gamma}d\nu, \tag{D1}$$

which is finite for $\alpha > 0$ and $\gamma > 0$. Integral (D1) can be decomposed into two regions, the first where $0 \leq \nu \leq 1$, in which

$$\nu^{\alpha-1-1/\gamma}(1-\exp(-\nu))^{1+1/\gamma} \approx \nu^\alpha\left(1-\frac{1}{2}\nu+\cdots\right)^{1+1/\gamma},$$

and the second where $1 \leq \nu < \infty$, on which the integrand can be approximated by $\nu^{\alpha-1-1/\gamma}$. The total integral can then be estimated by

$$\int_0^1 \nu^\alpha d\nu + \int_1^\infty \nu^{\alpha-1-1/\gamma}d\nu,$$

which gives

$$p_{\overline{N}}(0) \approx \frac{2\gamma\Gamma(2+1/\gamma)\alpha^{-\alpha}}{\pi\Gamma(\alpha)(1+\alpha)(1-\alpha\gamma)}(\overline{N})^{(1-\alpha\gamma)/\gamma},$$

as given in the text.

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